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Solution of a Nonlinear Integrodifferential System Arising in Nuclear Reactor Dynamics*

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1. INTRODUCTION

In a recent paper [8] Levin and Nohel have investigated the following nonlinear integrodifferential system arising in reactor dynamics.

$$\begin{aligned} u_t(t, x) &= u_{xx}(t, x) + \eta(x)\,(p(t) - p^*) \qquad (t > 0, -\infty < x < \infty),\\ C_i'(t) &= (\beta_i/\ell)\,p(t) - \lambda_i C_i\,, \qquad (t > 0) \quad i = 1,\dots, m,\\ p'(t) &= -p(t) \int_{-\infty}^{\infty} \omega(x)\, u(t, x)\, dx - (\beta/\ell)\,p(t) + \sum_{i=1}^{m} \lambda_i C_i(t) \qquad (t > 0), \end{aligned} \tag{0.1}$$

$$u(0, x) = \phi(x), \qquad C_i(0) = C_{i0} \quad (i = 1,\dots, m), \qquad p(0) = p_0\,, \tag{0.2}$$

where u is the incremental temperature from equilibrium, C_i is the concentration of the ith group delayed neutron precursor, p is the instantaneous power, and all the others are prescribed physical quantities. The term $\int_{-\infty}^{\infty} \omega(x)\, u(t, x)\, dx$ measures the incremental temperature feedback reactivity. (For more detailed physical meaning of these quantities see, e.g. [1]). In the system (0.1), (0.2) the reactor model is considered as an infinite rod and thus the temperature deviation u varies only along the rod. In actual reactor systems, however, the temperature is a function of position x, which may be one, two or three dimensional. Thus it is more realistic to consider the heat equation for u in a higher dimensional spatial domain (cf. [1]). In this paper we consider a more general system of integrodifferential equations in the form

$$\begin{aligned} u_t(t, x) &= Lu + f(t, x, u, p) \qquad (t \in (0, T], x \in \Omega),\\ C_i'(t) &= (\beta_i/\ell)\,p(t) - \lambda_i C_i(t) \qquad (t \in (0, T]), \quad i = 1,\dots, m,\\ p'(t) &= -g(q, p) - (\beta/\ell)\,p(t) + \sum_{i=1}^{m} \lambda_i C_i(t) \qquad (t \in (0, T]), \end{aligned} \tag{1.1}$$

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where Ω is an open domain in the euclidean space R^n, f is a (possibly nonlinear) continuous function of u, $p \in (-\infty, \infty)$, $t \in [0, T]$, $x \in \bar{\Omega}$ (the closure of Ω) and

$$Lu = \sum_{i,j=1}^{n} a_{ij}(t, x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(t, x)\, u_{x_i} + c(t, x)\, u,$$

$$g(q, p) = p(t)\, q(t) \equiv p(t) \int_\Omega \omega(x)\, u(t, x)\, dx.$$

The spatial domain Ω can either be bounded or unbounded. Thus if Γ denotes the boundary of Ω, then, in general, Γ consists of a bounded part Γ_b and an unbounded part Γ_∞. The boundary condition to be considered is given by

$$\begin{aligned} \alpha_1(t, x) \frac{\partial u}{\partial \nu} + \alpha_2(t, x)\, u &= 0 \qquad (t \in (0, T], x \in \Gamma_b), \\ \lim_{x \to \Gamma_\infty} u(t, x) &= 0 \qquad (t \in (0, T]), \end{aligned} \tag{1.2}$$

where ν is the outward normal unit vector on the bounded surface Γ_b, and α_1, α_2 are continuous functions on $[0, T] \times \Gamma_b$ such that $\alpha_1(t, x) \geqslant 0$, $\alpha_2(t, x) > 0$. The initial conditions are

$$u(0, x) = \phi(x), \qquad C_i(0) = C_{i0} \quad (i = 1,..., m), \qquad p_i(0) = p_0\,, \qquad (x \in \Omega). \tag{1.3}$$

It is to be mentioned that either the bounded boundary Γ_b or the unbounded part Γ_∞ of Γ can be empty. Thus, Ω is the whole space R^n if Γ_b is empty while it is a bounded domain, if Γ_∞ is empty. In this case only one of the boundary conditions in (1.2) appears. The general consideration includes the frequently discussed geometry in physical problems such as a half-space, an infinite or semiinfinite cylinder, an interior or exterior of a bounded medium, etc.

The purpose of this paper is to show the existence of a unique classical solution to the system (1.1)–(1.3) and to give a constructive method for the determination of the solution. Our approach to the problem is by successive approximations. In some special cases, which include the system (0,1), (0,2), explicit formulas for the calculation of the approximations are given.

Throughout this paper we always assume that the coefficients a_{ij}, b_i, c of L are continuous on $[0, T] \times \bar{\Omega}$ with $c(t, x) \leqslant 0$ and for some constant $a_0 > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(t, x)\, \xi_i \xi_j \geqslant a_0 \mid \xi \mid^2 \qquad ((t, x) \in [0, T] \times \bar{\Omega})$$

for every $\xi = (\xi_1, ..., \xi_n) \in R^n$. The initial function $\phi(x)$ is assumed continuous on $\bar{\Omega}$ and satisfying the boundary condition (1.2). For physical reasons, we also assume that $\omega(x)$, β, β_i, λ_i are nonnegative, ℓ is positive and

$$\bar{\omega} \equiv \int_\Omega \omega(x)\, dx < \infty, \qquad \beta = \sum_{i=1}^{m} \beta_i .$$

Concerning the function f, which may be nonlinear in u, p, we make the following assumptions.

(H_1) There exists positive constants p^*, ρ such that for every $(t, x) \in [0, T] \times \bar{\Omega}$,

$$f(t, x, u, p) > 0 \qquad \text{when} \qquad p > p^*, \qquad u \in (-\infty, \infty), \tag{1.4}$$

$$\begin{aligned} | f(t, x, u_1, p_1) - f(t, x, u_2, p_2)| \leqslant \rho(| u_1 - u_2 | + | p_1 - p_2 |) \\ (u_i, p_i \in (-\infty, \infty)). \end{aligned} \tag{1.5}$$

Notice that if $\eta(x)$ is bounded nonnegative on $\bar{\Omega}$ (which is physically the case), then the function

$$f(t, x, u, p) \equiv \eta(x)\,(p - p^*) \tag{1.6}$$

satisfies all the conditions in (H_1).

In case the effect of delayed neutrons is neglected then Eq. (1.1) reduces to

$$\begin{aligned} u_t(t, x) &= Lu + f(t, x, u, p), \\ p'(t) &= -g(q, p). \end{aligned} \tag{1.7}$$

The boundary condition remains the same as in (1.2) while the initial conditions become

$$u(0, x) = \phi(x), \qquad p(0) = p_0 . \tag{1.8}$$

We also discuss the existence and the construction of a solution for the system (1.7), (1.8), (1.2).

The integrodifferential system in a transformed form of (1.7), (1.8), (1.2) has been repeatedly investigated in recent years by Levin and Nohel [6, 7], Bronikowsi [2], Bronikowsi, Hall and Nohel [3], and Miller [9]; and many interesting results on the asymptotic behavior of a solution are obtained. In their work, however, the reactor model is always considered as either a finite or an infinite rod (i.e., Ω is an interval in R^1). On the other hand, if the function f is independent of p, then the first equation in (1.7) together with the boundary condition (1.2) and the initial condition $u(0, x) = \phi(x)$

becomes a classical initial-boundary value problem. This uncoupled nonlinear problem has been investigated by Chan [4] and by Sattinger [12] for a bounded spacial domain and by the author [10] for an arbitrary domain (i.e., bounded or unbounded) using the approach of successive approximations. The approach of the present paper for the coupled system (1.1)–(1.3) is closely related to that given in [10] and is motivated by the idea used in [11] for the treatment of a coupled system of Boltzmann equations arising in neutron transport problems. This approach involves successive approximations of some uncoupled linear initial-boundary problems.

The plan of the paper is as follows. In Section 2 we describe our approach to the problem (1.1)–(1.3) and show the existence of a unique solution to a "modified problem" by the method of successive approximations. It is shown in Section 3 that the solution of the modified problem is in fact the unique solution of the original problem. Section 4 deals with the existence and the construction of a solution for the problem (1.7), (1.8), (1.2).

2. Method of Successive Approximations

In this section we describe our approach to the problem by the method of successive approximations. Under some modifications of the third equation in (1.1) we prove an existence theorem for a "modified problem". The result of this theorem will be used to show the existence of a unique solution for the original problem (1.1)–(1.3) in the next section.

Let $J = [0, T]$, $D = (0, T] \times \Omega$ and $\bar{D} = [0, T] \times \bar{\Omega}$. Denote by $C(\bar{D})$ the Banach space of all bounded continuous functions $u(t, x)$ on $\bar{D}$ with norm

$$\| u \|_D = \sup\{| u(t, x)|; (t, x) \in \bar{D}\}.$$

Similarly denote by $C(\bar{\Omega})$, $C(J)$ the Banach space of all bounded continuous functions on $\bar{\Omega}$, J, respectively, with their norms given by

$$\| u \|_\Omega = \sup\{| u(x)|; x \in \bar{\Omega}\} \qquad (u \in C(\bar{\Omega})),$$
$$\| u \|_J = \sup\{| u(t)|; t \in J\} \qquad (u \in C(J)).$$

Let $\mathscr{X} = C(\bar{D}) \times C(J) \times \cdots \times C(J)$ be the product space, where $C(J) \times \cdots \times C(J)$ is taken $(m+1)$ times. With the usual definition of addition and scalar multiplication the space $\mathscr{X}$ equipped with the norm

$$\| U \|_{\mathscr{X}} = \| u_0 \|_D + \| u_1 \|_J + \cdots + \| u_{m+1} \|_J \qquad (U = (u_0, u_1, \ldots + u_{m+1}) \in \mathscr{X})$$

is also a Banach space.

We next transform the problem (1.1)–(1.3) by letting $u_0 = e^{-\gamma t} u$, $u_i = e^{-\gamma t} C_i$

$(i = 1,\ldots, m)$, $u_{m+1} = e^{-\gamma t} p$, where $\gamma > 0$ is a real constant to be chosen. Then Eq. (1.1) reduces to

$$\begin{aligned}
(u_0)_t - (L-\gamma)\, u_0 &= f_0(u_0, u_{m+1})\,(t, x) && ((t,x) \in D),\\
u_i' + \gamma u_i &= f_i(u_i, u_{m+1})\,(t), \quad i = 1,\ldots, m && (t \in (0, T]),\\
u_{m+1}' + \gamma u_{m+1} &= f_{m+1}(u_0, u_1, \ldots, u_{m+1})\,(t) && (t \in (0, T]),
\end{aligned} \tag{2.1}$$

and the boundary and initial conditions become

$$\begin{aligned}
\alpha_1(t, x) \frac{\partial u_0}{\partial \nu}(t, x) + \alpha_2(t, x)\, u_0(t, x) &= 0 \qquad (t \in (0, T], x \in \Gamma_b),\\
\lim_{x \to \Gamma_\infty} u_0(t, x) &= 0 \qquad (t \in (0, T]),
\end{aligned} \tag{2.2}$$

$$u_0(0, x) = \phi(x), \qquad u_i(0) = C_{i0} \quad (i = 1,\ldots, m), \qquad u_{m+1}(0) = p_0, \tag{2.3}$$

where

$$\begin{aligned}
f_0(u_0, u_{m+1})\,(t, x) &\equiv e^{-\gamma t} f(t, x, e^{\gamma t} u_0(t, x), e^{\gamma t} u_{m+1}(t)),\\
f_i(u_i, u_{m+1})\,(t) &\equiv -\lambda_i u_i(t) + (\beta_i/\ell)\, u_{m+1}(t), \qquad i = 1,\ldots, m,
\end{aligned}$$

$$f_{m+1}(u_0, u_1, \ldots, u_{m+1})\,(t) \equiv -e^{-\gamma t} g(e^{\gamma t} q_0(t), e^{\gamma t} p(t)) + \sum_{i=1}^{m} \lambda_i u_i(t) - (\beta/\ell)\, u_{m+1}(t). \tag{2.4}$$

In the last relation in (2.4), $q_0(t)$ is defined by

$$q_0(t) \equiv \int_\Omega \omega(x)\, u_0(t, x)\, dx.$$

Thus the existence and uniqueness problem of (1.1)–(1.3) follows from the same as for (2.1)–(2.3).

Consider the linear equation

$$(u_0)_t - (L - \gamma)\, u_0 = h(t, x) \qquad ((t, x) \in D) \tag{2.5}$$

together with the boundary condition (2.2) and the initial condition

$$u_0(0, x) = \phi(x) \qquad (x \in \Omega), \tag{2.6}$$

where h is a given function in $C(\bar{D})$. In order to describe our method of successive approximations we make the following assumptions.

(H_2) For some closed subset S of $C(\bar{D})$, f maps $S \times C(J)$ into S and the linear problem (2.5), (2.6), (2.2) has a unique solution $u_0 \in S$.

Remark 2.1. The existence of a solution to the linear problem (2.5), (2.6), (2.2) can be insured under sufficiently smooth condition on the boundary of Ω and some conditions of Hölder continuity on f, h, ϕ, α_1, α_2 and the coefficients of L, where S may be taken as the set of Hölder continuous functions (cf. [5, p. 320]). Furthermore, the closed property of S is used to insure that the limit function is also in S (see Remark 2.2).

Let $u_0^{(0)} \in S$, $u_i^{(0)} \in C(J)$ $(i = 1,..., m + 1)$ be given. Then by (H_2) we can construct a sequence $\{u_0^{(k)}(t, x), u_1^{(k)}(t),..., u_{m+1}^{(k)}(t)\}$ of functions in $\mathscr{X}$ from the following system.

$$
\begin{aligned}
(u_0^{(k)})_t - (L-\gamma)\, u^{(k)} &= f_0(u_0^{(k-1)}, u_{m+1}^{(k-1)})\,(t, x), \\
(u_i^{(k)})' + \gamma u_i^{(k)} &= f_i(u_i^{(k-1)}, u_{m+1}^{(k-1)})\,(t), \qquad (i = 1,..., m), \\
(u_{m+1}^{(k)})' + \gamma u_{m+1}^{(k)} &= f_{m+1}(u_0^{(k-1)}, u_1^{(k-1)},..., u_{m+1}^{(k-1)})\,(t),
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
\alpha_1(t, x)\frac{\partial u_0^{(k)}}{\partial \nu} + \alpha_2(t, x)\, u_0^{(k)} &= 0 \qquad (t \in (0, T], x \in \Gamma_b), \\
\lim_{x \to \Gamma_\infty} u_0^{(k)}(t, x) &= 0 \qquad (t \in (0, T]),
\end{aligned}
\tag{2.8}
$$

$$
u_0^{(k)}(0, x) = \phi(x), \qquad u_i^{(k)}(0) = C_{i0} \quad (i = 1,..., m), \qquad u_{m+1}^{(k)}(0) = p_0, \tag{2.9}
$$

where $k = 1, 2,...$. The process of construction is as follows. Starting with $k = 1$, we can find a unique function $u_0^{(1)}(t, x)$ satisfying the first equation in (2.7) together with the boundary condition (2.8) and the first initial condition in (2.9). This is possible since $f(u_0^{(0)}, u_{m+1}^{(0)})$ is known. For each $i = 1,..., m + 1$, we solve the uncoupled ordinary differential equation in (2.7), subject to the corresponding initial condition in (2.9) to obtain a solution $u_i^{(1)}(t)$. This is again possible since each f_i is a known function. Replacing $u_0^{(0)}, u_1^{(0)},..., u_{m+1}^{(0)}$ by the obtained functions $u_0^{(1)}, u_1^{(1)},..., u_{m+1}^{(1)}$ and continuing the same process we can determine a sequence $(u_0^{(k)}, u_1^{(k)},..., u_{m+1}^{(k)}\}$ satisfying the system (2.7)–(2.9) for every $k = 1, 2,...$. The aim of this section is to show that under the assumption (1.5) in (H_1) and a stronger condition on $g(q, p)$ the sequence $\{u_0^{(k)}, u_1^{(k)},..., u_{m+1}^{(k)}\}$ converges to a unique solution of the problem (2.1)–(2.3). To achieve this goal we first formulate the problem (2.1)–(2.3) as an operator equation in the Banach space $\mathscr{X}$.

Define operators A_0, A_i $(i = 1,..., m + 1)$ in $C(\bar{D})$ and $C(J)$, respectively, by

$$
\begin{aligned}
A_0 u_0 &= (u_0)_t - (L-\gamma)\, u_0 \qquad (u_0 \in D(A_0)), \\
A_i u_i &= u_i' + \gamma u_i, \qquad i = 1,..., m + 1 \qquad (u_i \in D(A_i)),
\end{aligned}
\tag{2.10}
$$

where for each $i = 0, 1, \ldots, m+1$, $D(A_i)$ is the domain of A_i defined by

$$\begin{aligned} D(A_0) &= \{u_0 \in C(\bar{D});\ (u_0)_t\,, Lu_0 \in C(\bar{D}) \text{ and } u_0 \text{ satisfies (2.2), (2.3)}\}, \\ D(A_i) &= \{u_i \in C(J);\ u_i' \in C(J) \text{ and } u_i(0) = C_{i0}\} \qquad (i = 1,\ldots, m), \\ D(A_{m+1}) &= \{u_{m+1} \in C(J);\ u_{m+1}' \in C(J) \text{ and } u_{m+1}(0) = p_0\}. \end{aligned} \tag{2.11}$$

Then A_0 is an operator with domain $D(A_0)$ and range $R(A_0)$ both in $C(\bar{D})$ and for each $i = 1,\ldots, m+1$, A_i is an operator with $D(A_i)$ and $R(A_i)$ both in $C(J)$. Next define operators $\mathscr{A}$, $\mathscr{F}$ in $\mathscr{X}$ by

$$\begin{aligned} \mathscr{A}U &= (A_0u_0\,, A_1u_1\,,\ldots, A_{m+1}u_{m+1}) \qquad (U = (u_0\,, u_1\,,\ldots, u_{m+1}) \in D(\mathscr{A})), \\ \mathscr{F}(U) &= (f_0(u_0\,, u_{m+1}), f_1(u_1\,, u_{m+1}),\ldots, f_m(u_m\,, u_{m+1}), f_{m+1}(u_0\,, u_1\,,\ldots, u_{m+1})) \\ &\qquad\qquad\qquad\qquad (U = (u_0\,, u_1\,,\ldots, u_{m+1}) \in \mathscr{X}), \end{aligned} \tag{2.12}$$

where $D(\mathscr{A})$ is the domain of $\mathscr{A}$ given by

$$D(\mathscr{A}) = D(A_0) \times D(A_1) \times \cdots \times D(A_{m+1}). \tag{2.13}$$

Then $\mathscr{A}$ is an operator with $D(\mathscr{A})$ and range $R(\mathscr{A})$ both in $\mathscr{X}$ and $\mathscr{F}$ maps the whole space $\mathscr{X}$ into itself. With this definition, the problem (2.1)–(2.3) becomes an operator equation

$$\mathscr{A}U = \mathscr{F}(U) \qquad (U \in D(\mathscr{A})) \tag{2.14}$$

in the Banach space $\mathscr{X}$. The requirement of U in $D(\mathscr{A})$ insures that the components $u_0\,, u_1\,,\ldots, u_{m+1}$ of U satisfy the boundary and initial conditions (2.2), (2.3). Hence, the proof of the existence of a solution to the problem (2.1)–(2.3) is reduced to show the existence of a function $U \in D(\mathscr{A})$ such that $\mathscr{A}U = \mathscr{F}(U)$ in $\mathscr{X}$. In order to accomplish this, we first prepare the following lemmas. For convenience, we set

$$\bar{c} = \sup\{c(t, x);\ (t, x) \in \bar{D}\}. \tag{2.15}$$

Notice that $\bar{c} \leqslant 0$ by hypothesis.

LEMMA 2.1. *For any $\gamma > 0$ and each $i = 1,\ldots, m+1$,*

$$\begin{aligned} \| A_0u_0 - A_0v_0 \|_D &\geqslant (\gamma - \bar{c}) \| u_0 - v_0 \|_D \qquad (u_0\,, v_0 \in D(A_0)), \\ \| A_iu_i - A_iv_i \|_J &\geqslant \gamma \| u_i - v_i \|_J \qquad (u_i\,, v_i \in D(A_i)). \end{aligned} \tag{2.16}$$

Moreover, for each $i = 0, 1,\ldots, m+1$, the inverse operator A_i^{-1} exists on $R(A_i)$ and

$$\begin{aligned} \| A_0^{-1}w_1 - A_0^{-1}w_2 \|_D &\leqslant (\gamma - \bar{c})^{-1} \| w_1 - w_2 \|_D \qquad (w_1\,, w_2 \in R(A_0)), \\ \| A_i^{-1}w_1 - A_i^{-1}w_2 \|_J &\leqslant \gamma^{-1} \| w_1 - w_2 \|_J \qquad (w_1\,, w_2 \in R(A_i)). \end{aligned} \tag{2.17}$$

Proof. We begin with the first inequality in (2.16). Let $u = u_0 - v_0$ and let (t_0, x_0) be any point in $\bar{D}$ such that $\| u \|_D = | u(t_0, x_0)|$. We first show that

$$u(t_0, x_0)(A_0 u_0 - A_0 v_0)(t_0, x_0) \geqslant (\gamma - c(t_0, x_0)) \, | u(t_0, x_0)|^2. \tag{2.18}$$

The above relation is trivial if $u(t_0, x_0) = 0$. We thus only consider the case $u(t_0, x_0) \neq 0$. Since $u(t_0, x_0)$ is either a positive maximum or a negative minimum on $\bar{D}$, the boundary and initial conditions (2.2), (2.3) imply that $t_0 \neq 0$ and $x_0 \notin \Gamma$. Notice that u satisfies the boundary condition (2.2) with $u(0, x) = 0$, since u_1, u_2 are in $D(A_0)$. Knowing $x_0 \in \Omega$, we have

$$\begin{gathered} u_{x_i}(t_0, x_0) = 0, \qquad (i = 1, \dots, n), \\ \sum_{i,j=1}^{n} a_{ij}(t_0, x_0) \, u_{x_i x_j}(t_0, x_0) \lesseqqgtr 0 \qquad \text{according to} \qquad u(t_0, x_0) \gtrless 0 \end{gathered} \tag{2.19}$$

(cf. [5, p. 34]). Since $t_0 \in (0, T]$, we also have

$$\begin{gathered} u_t(t_0, x_0) = 0 \qquad \text{if} \qquad t_0 \in (0, T), \\ u_t(T, x_0) \gtreqqless 0 \qquad \text{according to} \qquad u(T, x_0) \gtrless 0. \end{gathered} \tag{2.20}$$

But by the definition of A_0,

$$\begin{aligned} &u(t_0, x_0)(A_0 u_0 - A_0 v_0)(t_0, x_0) \\ &\quad = u(t_0, x_0) \Bigg[u_t(t_0, x_0) - \sum_{i,j=1}^{n} a_{ij}(t_0, x_0) \, u_{x_i x_j}(t_0, x_0) \\ &\qquad - \sum_{i=1}^{n} b_i(t_0, x_0) \, u_{x_i}(t_0, x_0) + (\gamma - c(t_0, x_0)) \, u(t_0, x_0) \Bigg]. \end{aligned} \tag{2.21}$$

We see from (2.19)–(2.21) that (2.18) holds. It follows from the relation

$$\| u \|_D \, \| A_0 u_0 - A_0 v_0 \|_D \geqslant u(t_0, x_0)(A_0 u_0 - A_0 v_0)(t_0, x_0) \geqslant (\gamma - \bar{c}) \, \| u \|_D^2$$

that the first inequality in (2.16) is proven.

To show the second inequality in (2.16), we observe from the definition of A_i that

$$u(t)(A_i u_i - A_i v_i) = (\tfrac{1}{2})(u^2(t))' + \gamma u^2(t) \qquad (t \in (0, T]),$$

and $u(0) = 0$, where $u(t) = u_i(t) - v_i(t)$. Hence, if $t_0 \in [0, T]$ is such that $\| u \|_J = | u(t_0)|$, then $(u^2(t))' \geqslant 0$ at $t = t_0$ and thus

$$u(t_0)(A_i u_i(t_0) - A_i v_i(t_0)) \geqslant \gamma u^2(t_0). \tag{2.22}$$

From the relation

$$\| u \|_J \, \| A_i u_i - A_i v_i \|_J \geqslant u(t_0)(A_i u_i(t_0) - A_i v_i(t_0)) \geqslant \gamma \, \| u \|_J^2,$$

we obtain the second relation in (2.16). Finally the existence of A_i^{-1} $(i = 0, 1, \ldots, m+1)$ and the relations in (2.17) follow directly from (2.16). This proves the lemma.

LEMMA 2.2. *For any* $\gamma > 0$,

$$\| \mathscr{A} U_1 - \mathscr{A} U_2 \|_{\mathfrak{X}} \geqslant \gamma \| U_1 - U_2 \|_{\mathfrak{X}} \qquad (U_1, U_2 \in D(\mathscr{A})). \tag{2.23}$$

Moreover, the inverse operator $\mathscr{A}^{-1}$ *exists and*

$$\| \mathscr{A}^{-1} W_1 - \mathscr{A}^{-1} W_2 \|_{\mathfrak{X}} \leqslant \gamma^{-1} \| W_1 - W_2 \|_{\mathfrak{X}} \qquad (W_1, W_2 \in R(\mathscr{A})). \tag{2.24}$$

Proof. Let $U_1 = \{u_0, u_1, \ldots, u_{m+1}\}$, $U_2 = \{v_0, v_1, \ldots, v_{m+1}\}$ be elements in $D(\mathscr{A})$. Then, by definition,

$$U_1 - U_2 = (u_0 - v_0, u_1 - v_1, \ldots, u_{m+1} - v_{m+1}),$$
$$\mathscr{A} U_1 - \mathscr{A} U_2 = (A_0 u_0 - A_0 v_0, A_1 u_1 - A_1 v_1, \ldots, A_{m+1} u_{m+1} - A_{m+1} v_{m+1}),$$

and by Lemma 2.1 we have

$$\begin{aligned} \| \mathscr{A} U_1 - \mathscr{A} U_2 \|_{\mathfrak{X}} &= \| A_0 u_0 - A_0 v_0 \|_D + \| A_1 u_1 - A_1 v_1 \|_J + \cdots \\ &\quad + \| A_{m+1} u_{m+1} - A_{m+1} v_{m+1} \|_J \\ &\geqslant \gamma (\| u_0 - v_0 \|_D + \| u_1 - v_1 \|_J + \cdots + \| u_{m+1} - v_{m+1} \|_J) \\ &= \gamma \| U_1 - U_2 \|_{\mathfrak{X}} . \end{aligned}$$

This proves (2.23). The existence of $\mathscr{A}^{-1}$ and the inequality (2.24) follows immediately from (2.23).

In order to prove the existence of a solution to (2.14), we require that each f_i $(i = 0, 1, \ldots, m+1)$ satisfies a global Lipschitz condition which is not fulfilled by the function f_{m+1} since $g(q, p)$ only satisfies a local Lipschitz condition. To overcome this difficulty, we choose some constants M_1, M_2 (to be determined later) and define a modification $\hat{g}$ so that $\hat{g}(q, p)$ satisfies a global Lipschitz condition while it coincides with $g(q, p)$ when $|q| \leqslant M_1$, $|p| \leqslant M_2$. Specifically, we define $\hat{g}(q, p)$ as follows.

$$\hat{g}(q,p) \equiv \begin{cases} g(q,p) & \text{if} \quad |q| \leqslant M_1, \quad |p| \leqslant M_2, \\ g(M_1, p) & \text{if} \quad q \geqslant M_1, \quad |p| \leqslant M_2, \\ g(-M_1, p) & \text{if} \quad q \leqslant -M_1, \quad |p| \leqslant M_2, \\ g(q, M_2) & \text{if} \quad |q| \leqslant M_1, \quad p \geqslant M_2, \\ g(q, -M_2) & \text{if} \quad |q| \leqslant M_1, \quad p \leqslant -M_2, \\ g(M_1, M_2) & \text{if} \quad q \geqslant M_1, \quad p \geqslant M_2, \\ g(M_1, -M_2) & \text{if} \quad q \geqslant M_1, \quad p \leqslant -M_2, \\ g(-M_1, M_2) & \text{if} \quad q \leqslant -M_1, \quad p \geqslant M_2, \\ g(-M_1, -M_2) & \text{if} \quad q \leqslant -M_1, \quad p \leqslant -M_1 . \end{cases} \tag{2.25}$$

The above definition of $\hat{g}$ shows that

$$|\hat{g}(q_1, p_1) - \hat{g}(q_2, p_2)| \leqslant M_1 |q_1 - q_2| + M_2 |p_1 - p_2| \tag{2.26}$$

for any $q_i, p_i \in (-\infty, \infty)$ $(i = 1, 2)$ which, in turn, implies that

$$\begin{aligned} &| e^{-\gamma t}\hat{g}(e^{\gamma t}q_1, e^{\gamma t}p_1) - e^{-\gamma t}\hat{g}(e^{\gamma t}q_2, e^{\gamma t}p_2)| \\ &\quad \leqslant e^{-\gamma t}[M_1 | e^{\gamma t}q_1 - e^{\gamma t}q_2 | + M_2 | e^{\gamma t}p_1 - e^{\gamma t}p_2 |] \\ &\quad = M_1 | q_1 - q_2 | + M_2 | p_1 - p_2 | . \end{aligned}$$

It follows from the above relation that for any $q_i, p_i \in C(J)$,

$$\begin{aligned} &\sup_{t\in J} | e^{-\gamma t}\hat{g}(e^{\gamma t}q_1(t), e^{\gamma t}p_1(t)) - e^{-\gamma t}\hat{g}(e^{\gamma t}q_2(t), e^{\gamma t}p_2(t))| \\ &\quad \leqslant M_1 \| q_1 - q_2 \|_J + M_2 \| p_1 - p_2 \|_J . \end{aligned} \tag{2.27}$$

Now define functions $\hat{f}_{m+1}$, $\hat{\mathscr{F}}$ by

$$\hat{f}_{m+1}(u_0, \dots, u_{m+1})(t) \equiv -e^{-\gamma t}\hat{g}(e^{\gamma t}q_0(t), e^{\gamma t}u_{m+1}(t)) + \sum_{i=1}^{n} \lambda_i u_i(t) - (\beta/\ell)\, u_{m+1}(t),$$

$$\hat{\mathscr{F}}(U) \equiv (f_0(u_0, u_{m+1}), f_1(u_1, u_{m+1}), \dots, f_m(u_m, u_{m+1}), \hat{f}_{m+1}(u_0, \dots, u_{m+1})). \tag{2.28}$$

Then we have the following.

Lemma 2.3. *Assume that the Condition* (1.5) *in* (H_1) *holds. Then* $\hat{\mathscr{F}}$ *satisfies the Lipschitz condition*

$$\| \hat{\mathscr{F}}(W_1) - \hat{\mathscr{F}}(W_2)\|_{\mathscr{X}} \leqslant K \| W_1 - W_2 \|_{\mathscr{X}} \qquad (W_1, W_2 \in \mathscr{X}), \tag{2.29}$$

where

$$K = \max\{\rho + \bar{\omega}M_1, \rho + M_2 + 2\beta/\ell, 2\lambda_1, \dots, 2\lambda_m\}. \tag{2.30}$$

Proof. Let $W_1 = (w_0, w_1, \dots, w_{m+1})$, $W_2 = (v_0, v_1, \dots, v_{m+1})$ be elements in $\mathscr{X}$. Then by the Condition (1.5) in (H_1),

$$| f_0(w_0, w_{m+1}) - f_0(v_0, v_{m+1})| \leqslant \rho(| w_0 - v_0 | + | w_{m+1} - v_{m+1} |),$$

which implies that

$$\| f_0(w_0, w_{m+1}) - f_0(v_0, v_{m+1})\|_D \leqslant \rho(\| w_0 - v_0 \|_D + \| w_{m+1} - v_{m+1} \|_J). \tag{2.31}$$

The linear property of f_i for $i = 1, \dots, m$ shows that

$$\| f_i(w_i, w_{m+1}) - f_i(v_i, v_{m+1})\|_J \leqslant \lambda_i \| w_i - v_i \|_J + (\beta_i/\ell) \| w_{m+1} - v_{m+1} \|_J . \tag{2.32}$$

In view of the Condition (2.27) and the definition of $\hat{f}_{m+1}$, we have

$$\begin{aligned} \|\hat{f}_{m+1}(w_0, w_1, \ldots, w_{m+1}) - \hat{f}_{m+1}(v_0, v_1, \ldots, v_{m+1})\|_J \\ \leqslant M_1 \| q_1 - q_2 \|_J + M_2 \| w_{m+1} - v_{m+1} \|_J \\ + \sum_{i=1}^{m} \lambda_i \| w_i - v_i \|_J + (\beta/\ell) \| w_{m+1} - v_{m+1} \|_J, \end{aligned} \tag{2.33}$$

where

$$q_1(t) = \int_\Omega \omega(x)\, w_0(t, x)\, dx, \qquad q_2(t) = \int_\Omega \omega(x)\, v_0(t, x)\, dx.$$

Since

$$\| q_1 - q_2 \|_J = \sup_{t \in J} \left| \int_\Omega \omega(x) [w_0(t, x) - v_0(t, x)]\, dx \right| \leqslant \bar{\omega} \| w_0 - v_0 \|_D, \tag{2.34}$$

we obtain from (2.33), (2.34) that

$$\begin{aligned} &\|\hat{f}_{m+1}(W_1) - \hat{f}_{m+1}(W_2)\|_J \\ &\quad \leqslant \bar{\omega} M_1 \| w_0 - v_0 \|_D + \sum_{i=1}^{m} \lambda_i \| w_i - v_i \|_J + (M_2 + \beta/\ell) \| w_{m+1} - v_{m+1} \|_J \end{aligned} \tag{2.35}$$

It follows from (2.31), (2.32) and (2.35) that

$$\begin{aligned} \| \hat{\mathscr{F}}(W_1) - \hat{\mathscr{F}}(W_2) \|_{\mathscr{X}} &\leqslant (\rho + \bar{\omega} M_1) \| w_0 - v_0 \|_D + \sum_{i=1}^{m} 2\lambda_i \| w_i - v_i \|_J \\ &\quad + (\rho + M_2 + 2\beta/\ell) \| w_{m+1} - v_{m+1} \|_J \\ &\leqslant K \| W_1 - W_2 \|_{\mathscr{X}}. \end{aligned}$$

This proves (2.29) and thus the lemma.

In the following theorem we show that if the function f_{m+1} is replaced by $\hat{f}_{m+1}$, then the problem (2.1)–(2.3), called the "modified problem" of (2.1)–(2.3), has a unique solution.

THEOREM 2.1. *Assume that* (H_2) *and the Condition* (1.5) *in* (H_1) ***hold***. *Then for any $\gamma > K$, the sequence $\{U^{(k)}\} = \{u_0^{(k)}, u_1^{(k)}, \ldots, u_{m+1}^{(k)}\}$ determined from the system* (2.7)–(2.9), *where f_{m+1} is replaced by $\hat{f}_{m+1}$, converges to a unique solution $U = (u_0, u_1, \ldots, u_{m+1})$ of the "modified problem"* (2.1)–(2.3). ***Furthermore***,

$$\| U^{(k)} - U \|_{\mathscr{X}} \leqslant \frac{K}{\gamma - K} \left(\frac{K}{\gamma} \right)^{k-1} \| U^{(1)} - U^{(0)} \|_{\mathscr{X}}, \qquad k = 1, 2, \ldots. \tag{2.36}$$

Proof. Let $W = (w_0, w_1, ..., w_{m+1})$ with $w_0 \in S$, $w_i \in C(J)$ $(i = 1, ..., m)$ be given. Then by (H_2) there exists $u_0 \in D(A) \cap S$ and $u_i \in D(A_i)$ such that

$$\begin{aligned} A_0 u_0 &= f_0(w_0, w_{m+1}), \\ A_i u_i &= f_i(w_i, w_{m+1}) \qquad (i = 1, ..., m), \\ A_{m+1} u_{m+1} &= \hat{f}_m(w_0, w_1, ..., w_{m+1}). \end{aligned}$$

The above relations are equivalent to the existence of a function

$$U = (u_0, u_1, ..., u_{m+1}) \in D(\mathscr{A})$$

such that $\mathscr{A} U = \hat{\mathscr{F}}(W)$. In view of Lemma 2.2, we may write $U = \mathscr{A}^{-1}\hat{\mathscr{F}}(W)$. Since for any $W_1, W_2 \in S \times \mathscr{C}(J)$, where $\mathscr{C}(J)$ is the product space of $C(J)$ taken $(m + 1)$–times, the results in Lemmas 2.2 and 2.3 imply that

$$\begin{aligned} \| \mathscr{A}^{-1}\hat{\mathscr{F}}(W_1) - \mathscr{A}^{-1}\hat{\mathscr{F}}(W_2)\|_{\mathscr{X}} &\leqslant \gamma^{-1} \| \hat{\mathscr{F}}(W_1) - \hat{\mathscr{F}}(W_2)\|_{\mathscr{X}} \\ &\leqslant \gamma^{-1} K \| W_1 - W_2 \|_{\mathscr{X}} . \end{aligned}$$

This shows that the operator $\mathscr{A}^{-1}\hat{\mathscr{F}}$ is a contraction mapping on $S \times \mathscr{C}(J)$. It follows from the contraction property of $\mathscr{A}^{-1}\hat{\mathscr{F}}$ that the sequence

$$\{U^k\} = \{u_0^{(k)}, u_1^{(k)}, ..., u_{m+1}^{(k)}\}$$

determined recursively from

$$U^{(k)} = \mathscr{A}^{-1}\hat{\mathscr{F}}(U^{(k-1)}), \qquad k = 1, 2, ..., \tag{2.37}$$

converges to a unique function $U \in S \times \mathscr{C}(J)$ such that $U = \mathscr{A}^{-1}\hat{\mathscr{F}}(U)$ and the error estimate (2.36) holds. This shows that $U \in D(\mathscr{A})$ and $\mathscr{A} U = \hat{\mathscr{F}}(U)$, which proves that the "modified problem" (2.1)–(2.3) has a unique solution $U = (u_0, u_1, ..., u_{m+1}) \in S \times \mathscr{C}(J)$. Since (2.37) is the operator equation for the system (2.7)–(2.9) except with f_{m+1} replaced by $\hat{f}_{m+1}$, the convergence of the sequence $\{u_0^{(k)}, u_1^{(k)}, ..., u_{m+1}^{(k)}\}$ determined from this system to a unique solution U of the modified problem (2.1)–(2.3) follows immediately. This completes the proof of the theorem.

Remark 2.2. In case the subset S is arbitrary (not necessarily closed) then the sequence $U^{(k)}$ determined from (2.37) converges to a unique element $U \in \mathscr{X}$ (u_0 may not be in S). In this situation we let $\bar{A}_0$ be the closure of A_0 in the sense that for any sequence $\{u^{(j)}\}$ in $D(A_0)$ with $u^{(j)} \to u_0$, $A_0 u^{(j)} \to w$ as $j \to \infty$, then $u_0 \in D(\bar{A}_0)$ and $\bar{A}_0 u_0 = w$. The operator $\bar{A}_0$ is unambiguously defined, that is, if $\{v^{(j)}\}$ is another sequence in $D(A_0)$ with $v^{(j)} \to u_0$, $A_0 v^{(j)} \to w^*$, then $w^* = w$. This follows from the usual definition of the closure of a closable linear operator, except in the present case the domain $D(A_0)$ is not a linear subspace when $\phi(x) \not\equiv 0$. Now let

$$\bar{\mathscr{A}} U = (\bar{A}_0 u_0, A_1 u_1, \ldots, A_{m+1} u_{m+1})$$

for $U \in D(\bar{\mathscr{A}})$, where

$$D(\bar{\mathscr{A}}) = D(\bar{A}_0) \times D(A_1) \times \cdots \times D(A_{m+1}).$$

Then it is easily seen from Lemma 2.1 that (2.16) holds for $\bar{A}_0$ and thus $\bar{\mathscr{A}}^{-1}$ exists and satisfies (2.24) with $\mathscr{A}^{-1}$ replaced by $\bar{\mathscr{A}}^{-1}$. Since $\mathscr{A} U^{(k)} = \hat{\mathscr{F}}(U^{(k-1)})$ and $U^{(k)} \to U$, $\hat{\mathscr{F}}(U^{(k-1)}) \to \hat{\mathscr{F}}(U)$ as $k \to \infty$, we have $U \in D(\bar{\mathscr{A}})$ and $\bar{\mathscr{A}} U = \hat{\mathscr{F}}(U)$. In this case the solution U is in the above extended sense.

From the definition of the functions u_i $(i = 0, 1, \ldots, m+1)$, we conclude that the function $\hat{U} \equiv (\hat{u}(t, x), \hat{C}_1(t), \ldots, \hat{C}_m(t), \hat{p}(t))$, where $\hat{u} = e^{\gamma t} u_0$, $\hat{C}_i = e^{\gamma t} u_i$ $(i = 1, \ldots, m)$, $\hat{p} = e^{\gamma t} u_{m+1}$, is the unique solution of the "modified problem" (1.1)–(1.3) is in the sense that U satisfies (1.1)–(1.3) with $g(q, p)$ replaced by $\hat{g}(q, p)$. Furthermore, by letting $u^{(k)} = e^{\gamma t} u_0^{(k)}$, $C_i^{(k)} = e^{\gamma t} u_i^{(k)}$, $p^{(k)} = e^{\gamma t} u_{m+1}^{(k)}$ for each $k = 1, 2, \ldots$, the sequence $\{u^{(k)}, C_1^{(k)}, \ldots, C_m^{(k)}, p^{(k)}\}$ converges to the unique solution $\hat{U}$. Since for each $k = 1, 2, \ldots$, the function $U^{(k)} = (u_0^{(k)}, u_1^{(k)}, \ldots, u_{m+1}^{(k)})$ satisfies the system (2.7)–(2.9), we see that the sequence $\{u^{(k)}, C_1^{(k)}, \ldots, C_m^{(k)}, p^{(k)}\}$ can be determined from the system

$$\begin{aligned} u_t^{(k)} - L u^{(k)} &= f(t, x, u^{(k-1)}, p^{(k-1)}), \\ (C_i^{(k)})' &= (\beta_i/\ell)\, p^{(k-1)} - \lambda_i C_i^{(k-1)}, \qquad i = 1, \ldots, m, \\ (p^{(k)})' &= -\hat{g}(q^{(k-1)}, p^{(k-1)}) - (\beta/\ell)\, p^{(k-1)} + \sum_{i=1}^{m} \lambda_i C_i^{(k-1)}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} \alpha_1(t, x) \frac{\partial u^{(k)}}{\partial \nu} + \alpha_2(t, x)\, u^{(k)} &= 0 \qquad (t \in (0, T],\ x \in \Gamma_b), \\ \lim_{x \to \Gamma_\infty} u^{(k)}(t, x) &= 0 \qquad (t \in (0, T]), \end{aligned} \tag{2.39}$$

$$u^{(k)}(0, x) = \phi(x), \qquad C_i^{(k)}(0) = C_{i0} \quad (i = 1, \ldots, m), \qquad p^{(k)}(0) = p_0, \tag{2.40}$$

for $k = 1, 2, \ldots$, where $q^{(k)}(t) = \int_\Omega \omega(x)\, u^{(k)}(t, x)\, dx$. This observation leads to the following conclusion.

THEOREM 2.2. *Assume that* (H_2) *and the Condition* (1.5) *in* (H_1) *hold. Then the sequence* $\{u^{(k)}, C_1^{(k)}, \ldots, C_m^{(k)}, p^{(k)}\}$ *determined from the system* (2.38)–(2.40) *converges to a unique solution* $\hat{U} = (\hat{u}, \hat{C}_1, \ldots, \hat{C}_m, \hat{p})$ *of the "modified problem"* (1.1)–(1.3) (*i.e., with* $g(q, p)$ *replaced by* $\hat{g}(q, p)$).

In the following section, we show that the function $\hat{U}$ given in Theorem 2.2 is in fact the unique solution of the original problem (1.1)–(1.3). Notice that if the Green's function of the uncoupled linear problem (2.5), (2.6), (2.2) is known, then the sequence $\{u^{(k)}, C_1^{(k)},..., C_m^{(k)}, p^{(k)}\}$ can be explicitly given. A simple example will also be given in the following section.

3. The Problem with Delayed Neutrons

In the previous section it is shown that the "modified problem" (1.1)–(1.3) (i.e., with $g(q, p)$ replaced by $\hat{g}(q, p)$) has a unique solution

$$\hat{U}(t, x) \equiv (\hat{u}(t, x), \hat{C}_1(t),..., \hat{C}_m(t), \hat{p}(t))$$

which can be constructed from the system (2.38)–(2.40). If we can show by a suitable choice of M_1, M_2 that this solution satisfies the condition

$$\left| \int_\Omega \omega(x)\, \hat{u}(t, x)\, dx \right| \leqslant M_1, \qquad |\hat{p}(t)| \leqslant M_2, \qquad (t \in [0, T]), \qquad (3.1)$$

then since $\hat{g}(q, p)$ coincides with $g(q, p)$ when $|q| \leqslant M_1$, $|p| \leqslant M_2$, the function $\hat{U}(t, x)$ is the unique solution of the original problem (1.1)–(1.3). Before proving (3.1), we establish some properties for a solution of the original problem. These properties will be needed in the determination of the constants M_1, M_2 in the definition of $\hat{g}(q, p)$.

Lemma 3.1. *If $(u(t, x), C_1(t),..., C_m(t), p(t))$ is a solution of the problem* (1.1)–(1.3) *then $p(t)$, $C_i(t)$ $(i = 1,..., m)$ are all positive on* $[0, T]$. *If, in addition, the condition* (1.4) *in* (H_1) *holds then $p(t)$, $C_i(t)$ are bounded on* $[0, T]$ *by some constant* K_2.

Proof. To show the first part of the lemma, we follow the same arguments given in [8]. Assume, by contradiction, that this were not the case. Then since $p(0) = p_0 > 0$, $C_i(0) = C_{i0} > 0$ $(i = 1,..., m)$, there exists $t_1 > 0$ such that $p(t) > 0$, $C_i(t) > 0$ for $t \in [0, t_1)$ and

$$\min\{p(t_1), C_1(t_1),..., C_m(t_1)\} = 0.$$

But from the second equation in (1.1),

$$\begin{aligned} C_i(t_1) &= C_{i0} \exp(-\lambda_i t_1) + (\beta_i/\ell) \int_0^{t_1} p(\tau) \exp(-\lambda_i(t_1 - \tau))\, d\tau \\ &> C_{i0} \exp(-\lambda_i t_1). \end{aligned}$$

We must have $p(t_1) = 0$. It follows from the third equation in (1.1) that

$$p'(t_1) = \sum_{i=1}^{m} \lambda_i C_i(t_1) > 0,$$

which together with $p(t) > 0$ on $[0, t_1)$ show that $p(t_1) = 0$ is impossible.

We next show that $p(t)$ is bounded on $[0, T]$. Assume, by contradiction, that $p(t)$ is unbounded. Then there exists $T_0 \in (0, T]$ such that $p(t) < \infty$ for $t \in [0, T_0)$ and $p(t) \to \infty$, as $t \to T_0$. Since $\beta = \sum_{i=1}^{m} \beta_i$, addition of the second and third equation in (1.1) gives

$$r'(t) = -g(q(t), p(t)) = -p(t) \int_\Omega \omega(x)\, u(t, x)\, dx, \tag{3.2}$$

where

$$r(t) \equiv p(t) + \sum_{i=1}^{m} C_i(t).$$

The above equation together with the unboundedness of $p(t)$ imply that

$$q^* \equiv \lim_{t \to T_0} \int_\Omega \omega(x)\, u(t, x)\, dx = -\infty. \tag{3.3}$$

To see this we observe from (3.2) that

$$r'(t) < 0 \qquad \text{at} \qquad t = T_0 \qquad \text{if} \qquad q^* > 0;$$

and for some finite Q,

$$|r'(t)| \leqslant Qp(t) \qquad \text{on} \qquad [0, T_0] \qquad \text{if} \qquad -\infty < q^* \leqslant 0.$$

The first inequality is impossible since $r(t) \to \infty$, as $t \to T_0$, and the second inequality is also impossible since it would imply that $p(t)$ is bounded on $[0, T_0]$. Therefore, (3.3) must hold. It follows from

$$\int_\Omega \omega(x)\, u(t, x)\, dx \geqslant \bar{\omega} \inf_{x \in \Omega}(u(t, x))$$

and (3.3) that

$$\lim_{t \to T_0}(\inf_{x \in \Omega} u(t, x)) = -\infty. \tag{3.4}$$

In view of the unboundedness of $p(t)$ at $t = T_0$ and the relation (3.4) we can find $t_0 \in (0, T_0)$ such that $p(t) > p^*$ and $\inf_{x \in \Omega}(u(t, x))$ is a negative decreasing function when $t \in [t_0, T_0)$. Let $t^* \in [t_0, T_0)$ be chosen and let x^* be a point in $\bar{\Omega}$ such that $u(t^*, x^*)$ is a negative minimum on $\bar{\Omega}$. (As will be shown in

Lemma 3.3 that $|u(t, x)|$ is bounded whenever $|p(t)|$ is bounded. This insures that $|u(t, x)| < \infty$ for all $(t, x) \in [0, t^*] \times \bar{\Omega})$. Then as in the proof of Lemma 2.1, we have $x^* \in \Omega$ and

$$u_{x_i}(t^*, x^*) = 0, \qquad \sum_{i,j=1}^{n} a_{ij}(t^*, x^*)\, u_{x_i x_j}(t^*, x^*) \geqslant 0. \tag{3.5}$$

Using the relation (3.5) and the first equation in (1.1), we have

$$u_t(t^*, x^*) \geqslant f(t^*, x^*, u(t^*, x^*), p(t^*)). \tag{3.6}$$

This is a contradiction since the left side of (3.6) is negative while the right side is positive in view of the Condition (1.4) in (H_1). This proves that $p(t)$ must be bounded on $[0, T]$. Finally from the second equation in (1.1) and the bounded property of $p(t)$ we conclude that $C_i(t)$ must also be bounded on $[0, T]$. This completes the proof of the lemma.

In order to show the bounded property of $u(t, x)$, we need the following lemma which is similar to that given in [10] but in an improved form.

LEMMA 3.2. *Let $u(t, x) \in C(\bar{D})$ and let (t_0, x_0) be any point in $[0, T) \times \Omega$ such that $u_t(t_0, x_0)$ exists. Then the right derivative of $|u(t, x_0)|$ exists at $t = t_0$ and*

$$|u(t_0, x_0)| \frac{d^+}{dt}(|u(t, x_0)|)_{t=t_0} = u(t_0, x_0)\, u_t(t_0, x_0). \tag{3.7}$$

Proof. The proof is similar to that given in [10]. Here we give a simplified proof in the present form as follows. For each $\delta > 0$, the inequality

$$\begin{aligned} &\delta^{-1} \big| [|u(t_0 + \delta, x_0)| - |u(t_0, x_0)|] - [|u(t_0, x_0) + \delta u_t(t_0, x_0)| - |u(t_0, x_0)|] \big| \\ &\quad \leqslant \delta^{-1} |u(t_0 + \delta, x_0) - u(t_0, x_0) - \delta u_t(t_0, x_0)| \end{aligned}$$

implies that

$$\frac{d^+}{dt}(|u(t, x_0)|)_{t=t_0} = \lim_{\delta \to 0^+} \delta^{-1}(|u(t_0, x_0) + \delta u_t(t_0, x_0)| - |u(t_0, x_0)|). \tag{3.8}$$

Since (3.7) holds for $|u(t_0, x_0)| = 0$, we need only to consider the case $|u(t_0, x_0)| \neq 0$. Now if $u(t_0, x_0) > 0$, then for sufficiently small δ,

$$u(t_0, x_0) + \delta u_t(t_0, x_0) > 0$$

and thus,

$$\delta^{-1}(|u(t_0, x_0) + \delta u_t(t_0, x_0)| - |u(t_0, x_0)|) = u_t(t_0, x_0). \tag{3.9}$$

Similarly if $u(t_0, x_0) < 0$ then,

$$\delta^{-1}(|u(t_0, x_0) + \delta u_t(t_0, x_0)| - |u(t_0, x_0)|) = -u_t(t_0, x_0). \tag{3.10}$$

It follows from (3.8)–(3.10) that

$$\frac{d^+}{dt}(|u(t, x_0)|)_{t=t_0} = \operatorname{sgn}(u(t_0, x_0))\, u_t(t_0, x_0) = \frac{u(t_0, x_0)}{|u(t_0, x_0)|}\, u_t(t_0, x_0),$$

which is equivalent to (3.7).

LEMMA 3.3. *If* $\{u(t, x), C_1(t), ..., C_m(t), p(t)\}$ *is a solution of the problem* (1.1)–(1.3) *and if* (H_1) *holds, then* $u(t, x)$ *is bounded on* $\bar{D}$ *by some constant* K_1.

Proof. Let $t_0 \in [0, T)$ be an arbitrary fixed point and let $x_0 \in \bar{\Omega}$ be a corresponding point such that $\| u(t_0)\|_\Omega = |u(t_0, x_0)|$. In view of the boundary condition (1.2), we have $x_0 \in \Omega$. By Lemma 3.2,

$$\begin{aligned} &|u(t_0, x_0)| \frac{d^+}{dt}(|u(t_0, x_0)|) \\ &\quad = u(t_0, x_0)\left[\sum_{i,j=1}^{n} a_{ij}(t_0, x_0)\, u_{x_i x_j}(t_0, x_0) + \sum_{i=1}^{n} b_i(t_0, x_0)\, u_{x_i}(t_0, x_0)\right. \\ &\qquad \left. + c(t_0, x_0)\, u(t_0, x_0) + f(t_0, x_0, u(t_0, x_0), p(t_0))\right]. \end{aligned}$$

With t_0 fixed, the argument in the proof of Lemma 2.1 shows that (2.19) still holds. Using the relation (2.19) in the above equation leads to

$$|u(t_0, x_0)|\,(d^+/dt)\,(|u(t_0, x_0)|) \leqslant |u(t_0, x_0)|\,|f(t_0, x_0, u(t_0, x_0), p(t_0))|. \tag{3.11}$$

Since by the hypothesis (1.5) with $u_2 = p_2 = 0$,

$$|f(t, x, u, p)| \leqslant |f(t, x, 0, 0)| + \rho(|u| + |p|),$$

and since by Lemma 3.1, $|p(t)| \leqslant K_2$ on $[0, T]$, we obtain from (3.11) that

$$|u(t_0, x_0)|\,(d^+/dt)\,(|u(t_0, x_0)|) < |u(t_0, x_0)|\,(\rho\,|u(t_0, x_0)| + K_0), \tag{3.12}$$

where K_0 is a constant independent of (t, x). Now if $\| u(t_0)\|_\Omega \neq 0$, we can divide (3.12) by $|u(t_0, x_0)|$ and write the resulting inequality in the form

$$\frac{d^+}{dt}\,[(|u(t_0, x_0)| + K_0\rho^{-1})\, e^{-\rho t_0}] < 0. \tag{3.13}$$

Notice that the usual rules of differentiation hold for right derivatives of a continuous function. The relation in (3.13) shows that the function $(| u(t, x_0)| + K_0\rho^{-1}) e^{-\rho t}$ is nonincreasing in some interval $[t_0, t_1]$ $(t_1 > t_0)$. Since this is true for every $t_0 \in [0, T)$ (and a corresponding $x_0 \in \Omega$ with $\| u(t_0)\|_\Omega = | u(t_0, x_0)|$) whenever $\| u(t_0)\|_\Omega \neq 0$, we conclude by starting from $t_0 = 0$ that

$$(\| u(t)\|_\Omega + K_0\rho^{-1}) e^{-\rho t} \leqslant \| \phi \|_\Omega + K_0\rho^{-1} \qquad (t \in [0, T)),$$

or equivalently,

$$\| u(t)\|_\Omega \leqslant e^{\rho t} \| \phi \|_\Omega + K_0\rho^{-1}(e^{\rho t} - 1) \qquad (t \in [0, T)). \tag{3.14}$$

This proves that $| u(t, x)|$ is bounded on $\bar{D}$ by K_1, where

$$K_1 \equiv e^{\rho T} \| \phi \|_\Omega + K_0\rho^{-1}(e^{\rho T} - 1).$$

With the results in Lemmas 3.1 and 3.3, we are now in a position to show the existence of a solution to the original problem.

THEOREM 3.1. *Assume that* (H_1), (H_2) *hold. Then by choosing* $M_1 > \bar{\omega} K_1$, $M_2 > K_2$ *in the definition of* $\hat{g}(q, p)$, *where* K_1, K_2 *are given in Lemmas* 3.1 *and* 3.3, *respectively, the solution* $\hat{U} = (\hat{u}, \hat{C}_1, \ldots, \hat{C}_m, \hat{p})$ *of the modified problem given in Theorem* 2.2 *is the unique solution of the original problem* (1.1)–(1.3).

Proof. By Theorem 2.2, the modified problem (1.1)–(1.3) has a unique solution $\hat{U} = (\hat{u}, \hat{C}_1, \ldots, \hat{C}_m, \hat{p})$. Since by definition $\hat{g}(q, p) = g(q, p)$ when $| q | \leqslant M_1$, $| p | \leqslant M_2$, it suffices to show that

$$| \hat{q}(t)| \equiv \left| \int_\Omega \omega(x)\, \hat{u}(t, x)\, dx \right| \leqslant M_1, \qquad | \hat{p}(t)| \leqslant M_2 \qquad (t \in [0, T]). \tag{3.15}$$

Suppose that (3.15) does not hold for some t in $[0, T]$. Then there exists a first $T^* \in (0, T]$ such that either $| \hat{q}(T^*)| = M_1$ or $| \hat{p}(T^*)| = M_2$. Consider the case where $| \hat{p}(T^*)| = M_2$ while $| \hat{q}(T^*)| \leqslant M_1$. By continuity, there exists $t^* \in (0, T^*)$ such that $K_2 + \epsilon \leqslant | \hat{p}(t)| \leqslant M_2$ and $| \hat{q}(t)| \leqslant M_1$ for $t \in [t^*, T^*]$, where $0 < \epsilon < M_2 - K_2$. This implies that $\hat{g}(\hat{q}, \hat{p}) = g(\hat{q}, \hat{p})$ and thus $\hat{q}(t)$, $\hat{p}(t)$ must satisfy the original problem (1.1)–(1.3) for $t \in (0, T^*]$. However, by Lemma 3.1, $\hat{p}(t)$ is bounded by K_2 and, in particular, $| \hat{p}(t)| \leqslant K_2$ for $t \in [t^*, T^*]$; we thus obtain a contradiction. In case T^* is the first value in $(0, T]$ such that $| \hat{q}(T^*)| = M_1$, then a similar argument as above using Lemma 3.3 instead of Lemma 3.1 also leads to a contradiction. This proves (3.15) and thus $\hat{U}$ is the unique solution of the original problem (1.1)–(1.3).

Remark 3.1. In case the nonlinear function $f(t, x, u, p)$ only satisfies a local Lipschitz condition in a neighborhood of (ϕ, p_0), then by defining a modification for f similar to that for g, the solution $\hat{U} = (\hat{u}, \hat{C}_1, ..., \hat{C}_m, \hat{p})$ of the corresponding modified problem is a "local solution" of the original problem for as long as $\{\hat{u}, \hat{p}\}$ remains in that neighborhood. Under some additional conditions on f it is possible to show that $\hat{U}$ is the unique solution of the original problem for all $(t, x) \in \bar{D}$. (For the case where f is independent of p, i.e., the system is uncoupled, see [10]).

Even though the modified solution $\hat{U}(t, x)$ given in Theorem 2.2 is the unique solution of the original problem, it does not necessarily mean that each approximation $U^{(k)}$ obtained from the system (2.38)–(2.40) is an approximate solution of the original problem. Therefore, in the construction of the solution $\hat{U}$ for the problem (1.1)–(1.3) it is sometimes necessary to use $\hat{g}(q, p)$ (rather $g(q, p)$) in the system (2.38)–(2.40). As we remarked earlier that if the Green's function of the corresponding linear problem is known, then the sequence $\{u^{(k)}, C_1^{(k)}, ..., C_m^{(k)}, p^{(k)}\}$ determined from the system (2.38)–(2.40) can be explicitly given. Consider, for example, the case where

$$Lu = \nabla^2 u = u_{x_1 x_1} + \cdots + u_{x_n x_n}, \qquad \Omega = R^n,$$

(so that Γ_b is empty). Then the solution of the equation

$$u_t - \nabla^2 u = h(t, x) \qquad (t, x) \in D, \tag{3.16}$$

subjecting to the boundary and initial conditions (1.2), (1.3) is given by

$$u(t, x) = \int_0^t \int_{R^n} G(t, x \mid \tau, \xi)\, h(\tau, \xi)\, d\xi\, d\tau + \int_{R^n} G(t, x \mid 0, \xi)\, \phi(\xi)\, d\xi, \tag{3.17}$$

where

$$G(t, x \mid \tau, \xi) = \frac{H(t - \tau)}{[4\pi(t - \tau)]^{n/2}} \exp\left(-\frac{|x - \xi|^2}{4(t - \tau)}\right), \tag{3.18}$$

and H is the Heaviside function. By letting

$$h(t, x) = f(t, x, u^{(k-1)}(t, x), p^{(k-1)}(t))$$

in (3.17), we obtain an explicit formula for the approximation $u^{(k)}$ which is given by

$$\begin{aligned} u^{(k)}(t, x) = &\int_0^t \int_{R^n} G(t, x \mid \tau, \xi) f(\tau, \xi, u^{(k-1)}(\tau, \xi), p^{(k-1)}(\tau))\, d\xi\, d\tau \\ &+ \int_{R^n} G(t, x \mid 0, \xi)\, \phi(\xi)\, d\xi, \qquad k = 1, 2, ... \end{aligned} \tag{3.19}$$

Since from the second and third equations in (2.38) the functions $C_i^{(k)}$ and $p^{(k)}$ are given by

$$C_i^{(k)}(t) = C_{i0} + (\beta_i/\ell) \int_0^t p^{(k-1)}(\tau)\, d\tau - \lambda_i \int_0^t C_i^{(k-1)}(\tau)\, d\tau, \qquad k = 1, 2, \ldots,$$
$$p^{(k)}(t) = p_0 - \int_0^t \left[\hat{g}(q^{(k-1)}(\tau), p^{(k-1)}(\tau)) - (\beta/\ell)\, p^{(k-1)}(\tau) + \sum_{i=1}^m \lambda_i C_i^{(k-1)}(\tau) \right] d\tau. \tag{3.20}$$

We see that the sequence $\{u^{(k)}, C_1^{(k)}, \ldots, C_m^{(k)}, p^{(k)}\}$ can be calculated from (3.19), (3.20) by straightforward integrations. Notice that in the formula (3.20) the function $\hat{g}$ (rather than g) is used. The above example includes the problem (0.1), (0.2) as a special case with $n = 1$,

$$f(t, x, u, p) = \eta(x)\, (p(t) - p^*).$$

4. The Problem Without Delayed Neutrons

In this section we discuss the existence and the construction of a solution for the system (1.7), (1.8), (1.2) by following the same approach as for the system (1.1)–(1.3). As in the previous section we assume that the hypotheses (H_1), (H_2) hold. Then by letting $u_0 = e^{-\gamma t} u$, $u_1 = e^{-\gamma t} p$, the system (1.7), (1.8), (1.2) becomes

$$\left.\begin{aligned} (u_0)_t - (L - \gamma)\, u_0 &= f_0(u_0, u_1)\,(t, x) \\ (u_1)' + \gamma u_1 &= f_1(u_0, u_1)\,(t) \end{aligned}\right\} \qquad ((t, x) \in D) \tag{4.1}$$

$$\begin{aligned} \alpha_1(t, x) \frac{\partial u_0}{\partial \nu} + \alpha_2(t, x)\, u_0 &= 0, \qquad (t \in (0, T], x \in \Gamma_b) \\ \lim_{x \to \Gamma_\infty} u(t, x) &= 0, \qquad (t \in (0, T]) \end{aligned} \tag{4.2}$$

$$u_0(0, x) = \phi(x), \qquad u_1(0) = p_0 \qquad (x \in \Omega), \tag{4.3}$$

where

$$\begin{aligned} f_0(u_0, u_1)\,(t, x) &\equiv e^{-\gamma t} f(t, x, e^{\gamma t} u_0(t, x), e^{\gamma t} u_1(t)), \\ f_1(u_0, u_1)\,(t) &\equiv -e^{-\gamma t} g(e^{\gamma t} q_0(t), e^{\gamma t} u_1(t)), \\ q_0(t) &\equiv \int_\Omega \omega(x)\, u_0(t, x)\, dx. \end{aligned} \tag{4.4}$$

Let $X = C(\bar{D}) \times C(J)$ be the underlying Banach space and define operators A, F in X by

$$\begin{aligned} AU &= (A_0 u_0, A_1 u_1) \qquad & (U = (u_0, u_1) \in D(A)), \\ F(U) &= (f_0(u_0, u_1), f_1(u_0, u_1)) \qquad & (U = (u_0, u_1) \in X), \end{aligned} \tag{4.5}$$

where A_0, A_1 are defined in (2.10) with $D(A_0)$, $D(A_1)$ given by (2.11) (for $m = 0$) and $D(A) = D(A_0) \times D(A_1)$. Then the system (4.1)–(4.3) is reduced to an operator equation

$$AU = F(U) \qquad (U \in D(A)) \tag{4.6}$$

in the Banach space X. Define $\hat{g}(q, p)$ by (2.25) and set

$$\begin{aligned} \hat{f}_1(u_0, u_1)(t) &= -e^{-\gamma t}\hat{g}(e^{\gamma t}q_0(t), e^{\gamma t}u_1(t)), \\ \hat{F}(U)(t, x) &= (f_0(u_0, u_1)(t, x), \hat{f}_1(u_0, u_1)(t)). \end{aligned} \tag{4.7}$$

Since for any $W_1 = (w_1, w_2)$, $W_2 = (v_1, v_2)$ in X,

$$\|\hat{f}_1(W_1) - \hat{f}_1(W_2)\|_J \leqslant \bar{\omega} M_1 \| w_1 - v_1 \|_D + M_2 \| w_2 - v_2 \|_J$$

in view of the modification $\hat{g}$, we have from (H_1) that

$$\begin{aligned} \|\hat{F}(W_1) - \hat{F}(W_2)\|_X &\leqslant (\rho + \bar{\omega} M_1) \| w_1 - v_1 \|_D + (\rho + M_2) \| w_2 - v_2 \|_J \\ &\leqslant K^* \| W_1 - W_2 \|_X, \end{aligned} \tag{4.8}$$

where

$$K^* = \max\{\rho_0 + \bar{\omega} M_1, \rho_0 + M_2\}. \tag{4.9}$$

On the other hand, from the Condition (2.17) in Lemma 2.1 and the proof of Lemma 2.2, it is easily seen that A^{-1} exists and

$$\| A^{-1}W_1 - A^{-1}W_2 \|_X \leqslant \gamma^{-1} \| W_1 - W_2 \|_X \qquad (W_1, W_2 \in R(A)). \tag{4.10}$$

The above inequality together with (4.8) implies that

$$\| A^{-1}\hat{F}(W_1) - A^{-1}\hat{F}(W_2)\|_X \leqslant \gamma^{-1}K^* \| W_1 - W_2 \|_X \qquad (W_1, W_2 \in S \times C(J))$$

This shows that for any choice of $\gamma > K^*$ in the definition of A_0, A_1 the operator $A^{-1}\hat{F}$ is a contraction mapping on $S \times C(J)$. Therefore, given $U^{(0)} = (u_0^{(0)}, u_1^{(0)}) \in S \times C(J)$ the sequence $\{U^{(k)}\} = \{u_0^{(k)}, u_1^{(k)}\}$ determined recursively from

$$U^{(k)} = A^{-1}\hat{F}(U^{(k-1)}) \qquad (k = 1, 2, \ldots), \tag{4.11}$$

converges to a unique function $U = (u_0, u_1) \in S \times C(J)$ such that $U = A^{-1}\hat{F}(U)$, that is, $U \in D(A)$ and $AU = \hat{F}(U)$. Since (4.11) is equivalent to the equation

$$\left.\begin{aligned} (u_0^{(k)})_t - (L - \gamma)\, u_0^{(k)} &= f_0(u_0^{(k-1)}, u_1^{(k-1)})(t, x) \\ (u_1^{(k)})' + \gamma u_1^{(k)} &= f_1(q_0^{(k-1)}, u_1^{(k-1)})(t) \end{aligned}\right\} \qquad k = 1, 2, \ldots, \tag{4.12}$$

subject to the boundary and initial conditions (2.8), (2.9) (with $m = 0$), we obtain the following conclusion.

THEOREM 4.1. *Assume that* (H_1), (H_2) *hold. Then for any* $\gamma > K^*$, *the sequence* $\{U^{(k)}\} = \{u_0^{(k)}, u_1^{(k)}\}$ *determined from* (4.12), (2.8), (2.9) (*with* $m = 0$) *converges to a unique solution* $U = (u_0, u_1)$ *of the modified problem* (4.1)–(4.3) (*i.e., with* f_1 *replaced by* $\hat{f}_1$). *Furthermore,*

$$\| U^{(k)} - U \|_X \leqslant [K^*/(\gamma - K^*)] (K^*/\gamma)^{k-1} \| U^{(1)} - U^{(0)} \|_X, \qquad k = 1, 2, \ldots \tag{4.13}$$

An immediate consequence of Theorem 4.1 is the following.

THEOREM 4.2. *Assume that* (H_1), (H_2) *hold. Then the sequence* $\{u^{(k)}, p^{(k)}\}$ *determined from the system*

$$\left.\begin{aligned} u_t^{(k)} - Lu^{(k)} &= f(t, x, u^{(k-1)}, p^{(k-1)}) \\ (p^{(k)})' &= -\hat{g}(q^{(k-1)}, p^{(k-1)}) \end{aligned}\right\} \qquad k = 1, 2, \ldots, \tag{4.14}$$

under the boundary and initial conditions (2.39), (2.40) (*with* $m = 0$) *converges to a unique solution* $(\hat{u}, \hat{p})$ *of the "modified problem"* (1.7), (1.8), (1.2) (*i.e., with* g *replaced by* $\hat{g}$).

To show the existence of a solution for the original problem (1.7), (1.8), (1.2) we need to choose some constants M_1, M_2 in the definition of $\hat{g}(q, p)$ and show that the solution $(\hat{u}, \hat{p})$ given in Theorem 4.2 satisfies the condition $| \int_\Omega \omega(x) \hat{u}(t, x)\, dx | \leqslant M_1$, $|\hat{p}(t)| \leqslant M_2$ for $t \in [0, T]$. If this can be done, then $\{\hat{u}, \hat{p}\}$ is the unique solution of the original problem. To accomplish this we observe from the second equation in (1.7) that if $(u(t, x), p(t))$ is a solution of (1.7), (1.8), (1.2) then,

$$p(t) = p_0 \exp\left(-\int_0^t q(s)\, ds\right) > 0,$$

and thus the existence of constants K_1, K_2 such that $| u(t, x)| \leqslant K_1$, $|p(t)| \leqslant K_2$ for $(t, x) \in \bar{D}$ follows from the same arguments as in the proofs of Lemmas 3.1 and 3.3. This observation leads to the following existence theorem for the problem (1.7), (1.8), (1.2).

THEOREM 4.3. *Assume that* (H_1), (H_2) *hold. Then by choosing* $M_1 > \bar{\omega} K_1$, $M_2 > K_2$, *the solution* $\{\hat{u}, \hat{p}\}$ *given in Theorem* 4.2 *is the unique solution of the original problem* (1.7), (1.8), (1.2).

Proof. Since $\hat{g}(u, p) = g(u, p)$ whenever $|u| \leqslant M_1$, $|p| \leqslant M_2$, it suffices to show that $|\hat{u}(t, x)| \leqslant M_1$, $|\hat{p}(t)| \leqslant M_2$ for $(t, x) \in \bar{D}$. But this follows from the same argument as in the proof of Theorem 3.1.

The results in Theorems 4.2 and 4.3 insure that if the constants M_1, M_2 in the definition of $\hat{g}(q, p)$ are chosen sufficiently large, then the sequence

$\{u^{(k)}, p^{(k)}\}$ determined from (4.14), (2.39), (2.40) converges to a unique solution of the problem (1.7), (1.8), (1.2). In the special case of $Lu = \nabla^2 u$, $\Omega = R^n$ this sequence can be constructed from the formula (3.19) for the approximations $u^{(k)}$ and the formula

$$p^{(k)}(t) = p_0 - \int_0^t \hat{g}(q^{(k-1)}(\tau), p^{(k-1)}(\tau))\, d\tau \qquad (k = 1, 2, \ldots), \tag{4.15}$$

for the approximations $p^{(k)}$.

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